Trees contained in every orientation of graphs

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Let K be a complete graph on k vertices, let G be a graph with chromatic number k, and let T be an oriented tree. If every orientation of K contains T, must every orientation of G contain T as well? I conjecture that this is true. This is known to hold for directed paths, paths with at most one 'change in direction', and for every oriented star.

To make things more precise, we introduce the following notation. Let T be an oriented tree on n vertices. We define $t(T)$ as the smallest integer k such that every orientation of the complete graph on k vertices contains T, and define $q(T)$ as the smallest integer k such that for every graph G with chromatic number k, every orientation of G contains T.

El Sahili proved that $t(T) \leq 3n-3$ for all T; Kühn, Mycroft and Osthus proved that $t(T) \leq 2n-2$ for all T if n is sufficiently large—this bound is best possible if no constraints are put on T; Mycroft and Naia proved that $t(T) = n$ for almost every tree T. Burr proved that $q(T) \leq (n-1)^2$ and Addario-Berry, Havet, Sales, Reed and Thomassé proved that $q(T) \leq \binom{n}{2}+1$; Gallai, Hasse, Roy and Vitaver each (independently) proved that if T is a directed path then $q(T) = n$; Addario-Berry, Havet and Thomassé proved that $q(T) = n$ when T is a path with two blocks. Clearly, $n \leq t(T) \leq q(T)$ holds for every tree, and the conjecture above is equivalent to the statement

$$t(T) = q(T) \text{ for every oriented tree } T.$$

Note that if $q(T) = n$ then the conjecture holds for T. Therefore it holds for paths with at most two blocks (i.e., with at most one change in direction). It can also be shown to hold for every orientation of a star (when the common value of $t(T)$ and $q(T)$ lies in $\{2n-2, 2n-3\}$).